THE NORMAL GRAVITY FIELD

The NORMAL gravity field is a REFERENCE SURFACE for the external gravity field of the Earth.

The source of the normal gravity field is a MODEL EARTH which best fits the actual shape of the Earth. An ELLIPSOID (an ellipse of revolution) is assumed for the model Earth and this ellipsoid is said to have the same mass (M) as the Earth, the same angular velocity (ω - omega) and the surface of this ellipsoid is said to be a LEVEL SURFACE (equipotential surface) of its own gravity field.

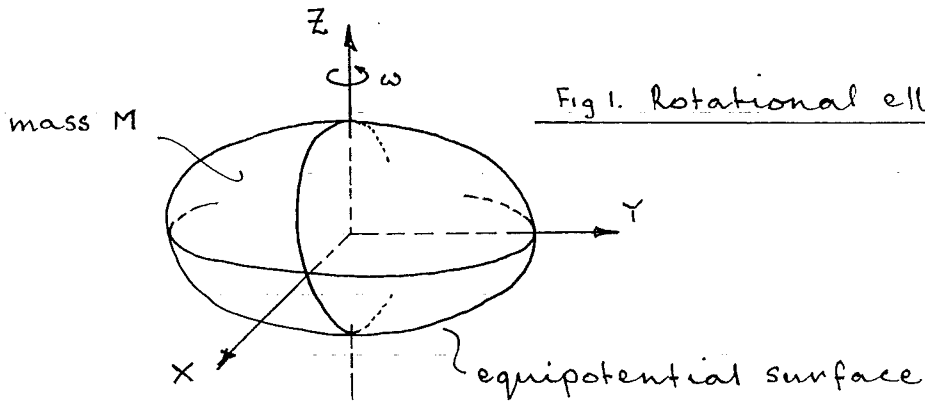

Fig 1. Rotational ellipsoid

(i) The gravity potential (U) at a point on this ellipsoid will be

$$U = V + \Phi \qquad \cdots (1)$$

U: normal gravity potential

V: potential due to mass attraction

Φ: centrifugal potential

or

$$U = V + \frac{1}{2}\omega^2 (X^2 + Y^2) \qquad \cdots (2)$$

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(i) cont'.

The gravitational potential V satisfies Laplace's equation ($\nabla^2 V = 0$) in the space exterior to the ellipsoid (semi-major axis a, flattening f)

(ii) The normal gravity field is to be rotationally symmetric

(iii) The level surface of the normal ellipsoid (U_o) is to have the same gravity potential as the geoid (W_o) hence

$$U_o = W_o \qquad \text{--- (3)}$$

level surface of model Earth (U_o); level surface of Earth (W_o)

To develop an expression for POTENTIAL due to CENTRIFUGAL FORCE (or Rotational potential) for a point on an ellipsoid of revolution, a SINGLE PARAMETER ELLIPSOIDAL COORDINATE SYSTEM is required

Consider an ellipsoid whose parameters are a (semi-major axis) and b (semi-minor axis)

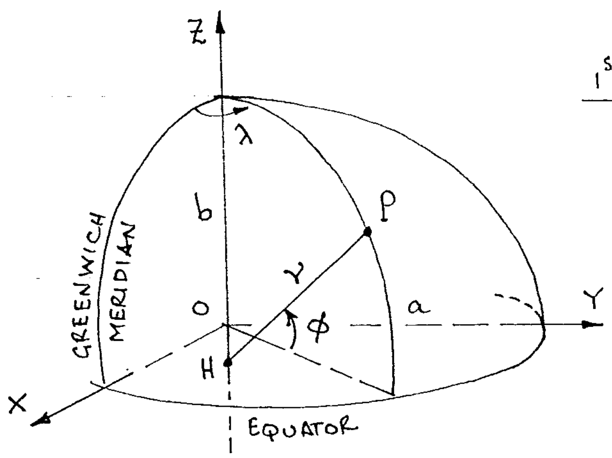

Fig. 2 Ellipsoid of revolution

1st eccentricity squared e^2

$$e^2 = \frac{a^2 - b^2}{a^2} \qquad \text{--- (4a)}$$

flattening f

$$f = \frac{a - b}{a} \qquad \text{--- (4b)}$$

and with ϕ = geodetic latitude, λ = longitude the following relationships are standard

$$e^2 = f(2-f) \qquad \text{.... (4c)}$$

$$b = a(1-f) \qquad \text{.... (4d)}$$

$$OH = \nu e^2 \sin\phi$$

$$\nu = \frac{a}{(1-e^2\sin^2\phi)^{1/2}}$$ = radius of curvature in prime vertical plane

... (4e)

Cartesian coords are

$$X = \nu\cos\phi\cos\lambda$$

$$Y = \nu\cos\phi\sin\lambda$$

$$Z = \nu(1-e^2)\sin\phi \qquad \text{.... (5)}$$

Now consider a FAMILY OF ELLIPSOIDS of constant focal length E whose semi-minor axis is u and semi-major axis is $\sqrt{E^2+u^2}$

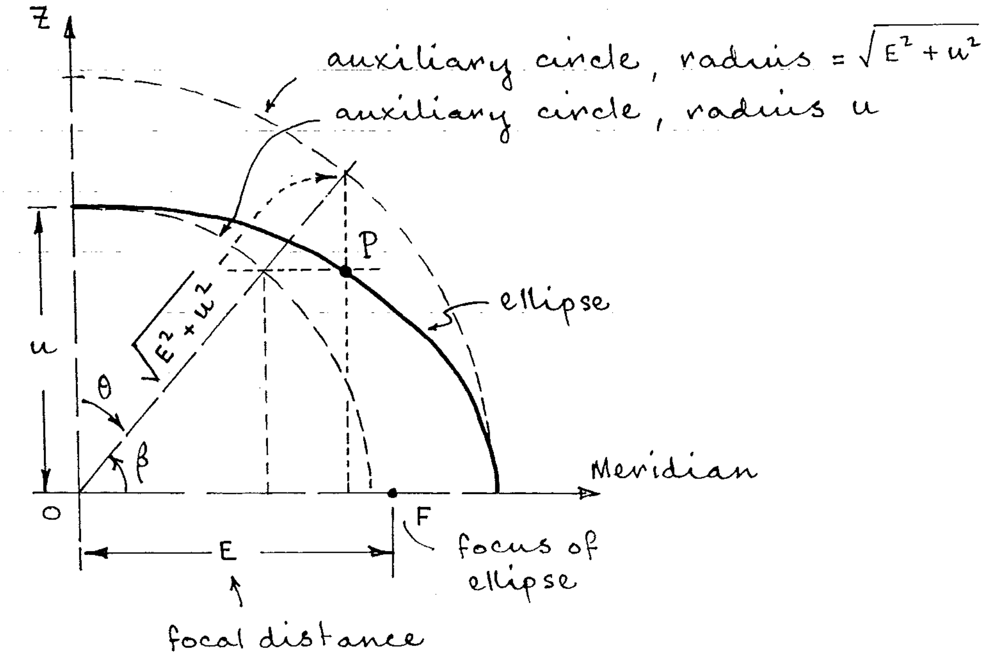

Fig. 3 Ellipse and auxiliary circles

β = "REDUCED" LATITUDE (or PARAMETRIC LATITUDE)

$\theta = 90 - \beta$

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By definition, the eccentricity $e = \dfrac{E}{\sqrt{E^2+u^2}}$ ... (6)

and for the normal ellipsoid

$$u = b$$
$$E^2 + b^2 = a^2 \qquad \text{... (7)}$$
$$E = \sqrt{a^2-b^2}$$

Cartesian coords are then

$$X = \sqrt{E^2+u^2}\,\sin\theta\,\cos\lambda$$
$$Y = \sqrt{E^2+u^2}\,\sin\theta\,\sin\lambda$$
$$Z = u\cos\theta \qquad \text{.... (8)}$$

and for a family of ellipsoids with the same focal length E, ELLIPSOIDAL COORDINATES are (u, θ, λ)

semi-minor axis — polar angle — longitude

Note that when $E=0$ (a sphere) the usual SPHERICAL COORDS $(u=r, \theta, \lambda)$ are the limiting case and also that u, θ, λ are ORTHOGONAL CURVILINEAR COORDINATES.

Now $V(u,\theta,\lambda)$ – a scalar function of position – satisfies Laplace's equation, $\nabla^2 V = 0$ in the space exterior to the ellipsoid. Laplace's equation in ELLIPSOIDAL COORDINATES (u,θ,λ) is

$$\boxed{(E^2+u^2)\frac{\partial^2 V}{\partial u^2} + 2u\frac{\partial V}{\partial u} + \frac{\partial^2 V}{\partial \theta^2} + \cot\theta\,\frac{\partial V}{\partial \theta} + \frac{E^2\cos^2\theta + u^2}{(E^2+u^2)\sin^2\theta}\,\frac{\partial^2 V}{\partial \lambda^2} = 0} \qquad \text{... (9)}$$

(see Heiskanen & Moritz, (1967), p. 41, eq. (1–105'))

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Note that when $E = 0$, $u = r$ and (9) becomes

$$\underbrace{r^2 \frac{\partial^2 V}{\partial r^2} + 2r \frac{\partial V}{\partial r}} + \underbrace{\frac{\partial^2 V}{\partial \theta^2} + \cot\theta \frac{\partial V}{\partial \theta}} + \frac{r^2}{r^2 \sin^2\theta} \frac{\partial^2 V}{\partial \lambda^2} = 0$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta} \right) + \frac{r^2}{r^2 \sin^2\theta} \frac{\partial^2 V}{\partial \lambda^2} = 0$$

and dividing through by r^2 gives Laplace's equation in spherical coordinates (r, θ, λ)

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 V}{\partial \lambda^2} = 0$$

To solve equation (9), consider the potential V as the product of three functions of the form

$$V(u, \theta, \lambda) = f(u)\, g(\theta)\, h(\lambda) \qquad \text{--- (10)}$$

where $f = f(u)$ is a function of the semi-minor axis u

$g = g(\theta)$ " " " " " polar angle θ only

$h = h(\lambda)$ " " " " " longitude only.

and by using methods set out in Heiskanen & Moritz (1967, pp. 41-42) the solution of Laplace's eqn $\nabla^2 V = 0$ is

$$(E^2 + u^2) \frac{\partial^2 f}{\partial u^2} + 2u \frac{\partial f}{\partial u} - \left\{ n(n+1) - \frac{E^2}{E^2 + u^2} m^2 \right\} f = 0 \qquad \text{... (11)}$$

$$\sin^2\theta \frac{\partial^2 g}{\partial \theta^2} + \cos\theta \sin\theta \frac{\partial g}{\partial \theta} + \left\{ n(n+1) \sin^2\theta - m^2 \right\} g = 0 \qquad \text{... (12)}$$

$$\frac{\partial^2 h}{\partial \lambda^2} + m^2 h = 0 \qquad \text{... (13)}$$

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Equations (11) and (12) can be reduced to a standard form by the substitutions, firstly for eqn (11)

$$z = i\frac{u}{E} \qquad \text{where} \quad i^2 = -1 \quad \text{or} \quad i = \sqrt{-1} \qquad \cdots (14)$$

and

$$1 - z^2 = 1 + \frac{u^2}{E^2} = \frac{E^2 + u^2}{E}$$

giving
$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial u} = \frac{i}{E}\frac{\partial f}{\partial z}$$

$$\frac{\partial^2 f}{\partial u^2} = -\frac{1}{E^2}\frac{\partial^2 f}{\partial z^2}$$

and secondly for eqn (12)

$$t = \cos\theta \qquad \cdots\cdots (15)$$

giving
$$\frac{\partial g}{\partial \theta} = \frac{\partial g}{\partial t} \cdot \frac{\partial t}{\partial \theta} = -\sin\theta \frac{\partial g}{\partial t}$$

$$\frac{\partial^2 g}{\partial \theta^2} = (1 - t^2)\frac{\partial^2 g}{\partial t^2} - t\frac{\partial g}{\partial t}$$

Making these substitutions leads to the solution of Laplace's eqn $\nabla^2 V = 0$ as

$$\boxed{(1 - z^2)\frac{\partial^2 f}{\partial z^2} - 2z\frac{\partial f}{\partial z} + \left\{ n(n+1) - \frac{m^2}{1 - z^2} \right\} f = 0} \qquad \cdots (16)$$

$$\boxed{(1 - t^2)\frac{\partial^2 g}{\partial t^2} - 2t\frac{\partial g}{\partial t} + \left\{ n(n+1) - \frac{m^2}{1 - t^2} \right\} g = 0} \qquad \cdots (17)$$

$$\boxed{\frac{\partial^2 h}{\partial \lambda^2} + m^2 h = 0} \qquad \cdots (18)$$

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The solution of (18) is

$$h(\lambda) = \sum_{m=0}^{\infty} (C \cos m\lambda + D \sin m\lambda)$$

The solution of (17) is

$$g(t) = P_n^m(t)$$

where $t = \cos\theta$ and $-1 < t < 1$ and $P_n^m(t)$ is the Associated Legendre function

The solution of (16) is

$$f(z) = Q_n^m(z)$$

where z is a complex number $z = \frac{iu}{E}$

and $Q_n^m(z)$ is an Associated Legendre function of the Second Kind.

Note that Legendre functions of the second kind as solutions to Legendres differential equation when the argument is a complex variable

Special results ($m=0$) that will be used at a later stage are

$$Q_0(z) = \coth^{-1} z$$

$$Q_2(z) = \left(\frac{3}{2}z^2 - \frac{1}{2}\right)\coth^{-1} z - \frac{3}{2} z$$

and

$$\coth^{-1} z = \coth^{-1}(ix) = -i \cot^{-1} x = -i \tan^{-1} \frac{1}{x}$$

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combining the solutions to (16), (17) and (18) gives

$$V(u,\theta,\lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} Q_n^m\left(i\frac{u}{E}\right)\left\{P_n^m(t)\left(a_n^m \cos m\lambda + b_n^m \sin m\lambda\right)\right\} \quad \text{--- (19)}$$

where $t = \cos\theta$

$z = i\dfrac{u}{E}$, $i^2 = -1$

as the potential <u>external</u> to an ellipsoid of semi-minor axis u and focal length E. n and m are integers (degree and order respectively) and a_n^m, b_n^m are coefficients.

In order to make all coefficients a_n^m and b_n^m real, division by $Q_n^m(i\frac{b}{E})$ is allowed where b is the semi-minor axis of an arbitrary but fixed <u>REFERENCE ELLIPSOID</u>. Hence

$$V(u,\theta,\lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{Q_n^m\left(i\frac{u}{E}\right)}{Q_n^m\left(i\frac{b}{E}\right)}\left\{P_n^m(t)\left(a_n^m \cos m\lambda + b_n^m \sin m\lambda\right)\right\} \quad \text{--- (20)}$$

Now, since the potential V must be rotationally symmetric (a requirement of a reference gravity field) then all terms containing $m\lambda$ will be zero, ie all non-zonal terms will be zero, thus

$$\boxed{V(u,\theta) = \sum_{n=0}^{\infty} \frac{Q_n\left(\frac{iu}{E}\right)}{Q_n\left(\frac{ib}{E}\right)} A_n P_n(\sin\beta)} \quad \text{--- (21)}$$

where $t = \cos\theta = \sin\beta$ and $P_n(t) = P_n(\cos\theta) = P_n(\sin\beta)$ and $\beta = 90 - \theta$ is the REDUCED LATITUDE. (or PARAMETRIC LATITUDE)

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The normal gravity potential U is given by (1) as

$$U = V + \Phi$$

where the rotational potential (or centrifugal potential) is given as

$$\Phi = \frac{1}{2}\omega^2 (X^2 + Y^2)$$

Using (8) with $u = b$ (ie the REFERENCE ELLIPSOID) and replacing $\sin\theta$ with $\cos\beta$ (since $\beta = 90 - \theta$)

$$\begin{aligned} X^2 + Y^2 &= (E^2 + b^2)\cos^2\beta \cos^2\lambda + (E^2 + b^2)\cos^2\beta \sin^2\lambda \\ &= (E^2 + b^2)\cos^2\beta \end{aligned}$$

hence the rotational potential Φ is

$$\boxed{\Phi = \frac{1}{2}\omega^2 (X^2 + Y^2) = \frac{1}{2}\omega^2 (E^2 + b^2)\cos^2\beta} \quad \text{---- (22)}$$

Combining (21) and (22) gives the normal gravity potential <u>on the surface of the reference ellipsoid</u> <u>$u = b$</u> as

$$\boxed{U_0 = V_0 + \Phi = \sum_{n=0}^{\infty} A_n P_n(\sin\beta) + \frac{1}{2}\omega^2 (E^2 + b^2)\cos^2\beta} \quad \text{--- (23)}$$

and the normal gravity potential at points <u>exterior to the surface</u> as

$$\boxed{U = V + \Phi = \sum_{n=0}^{\infty} \frac{Q_n\left(\frac{iu}{E}\right)}{Q_n\left(\frac{ib}{E}\right)} A_n P_n(\sin\beta) + \frac{\omega^2}{2}(E^2 + u^2)\cos^2\beta} \quad \text{--- (24)}$$

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TO DETERMINE EXPRESSIONS FOR THE COEFFICIENTS A_n

$$U_0 = \sum_{n=0}^{\infty} A_n P_n(\sin\beta) + \frac{\omega^2}{2}(E^2 + b^2)\cos^2\beta \qquad \cdots (23)$$

Refering to Fig. 3 and also eq (7)

$$E^2 + b^2 = a^2 \qquad \cdots (a)$$

and from the Legendre polynomials some values are

$$P_0(t) = 1$$
$$P_1(t) = t$$
$$P_2(t) = \tfrac{1}{2}(3t^2 - 1) \qquad \text{where} \quad t = \cos\theta = \sin\beta$$

hence

$$\begin{aligned} P_2(t) &= \tfrac{1}{2}(3\sin^2\beta - 1) \\ &= \tfrac{1}{2}(3(1-\cos^2\beta) - 1) \\ &= 1 - \tfrac{3}{2}\cos^2\beta \end{aligned}$$

or

$$\cos^2\beta = \tfrac{2}{3}\left[1 - P_2(t)\right] = \tfrac{2}{3}\left[1 - P_2(\sin\beta)\right] \qquad \cdots (b)$$

Substituting (a) and (b) into (23) and re-arranging gives

$$\sum_{n=0}^{\infty} A_n P_n(\sin\beta) + \tfrac{1}{3}\omega^2 a^2 - \tfrac{1}{3}\omega^2 a^2 P_2(\sin\beta) - U_0 = 0 \qquad \cdots (25)$$

Expanding (25) with $t = \sin\beta$ gives

$$A_0 P_0(t) + A_1 P_1(t) + A_2 P_2(t) + \sum_{n=3}^{\infty} A_n P_n(t)$$

$$+ \tfrac{1}{3}\omega^2 a^2 - \tfrac{1}{3}\omega^2 a^2 P_2(t) - U_0 = 0$$

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and since $P_0(t) = 1$ we may write

$$\left(A_0 + \tfrac{1}{3}\omega^2 a^2 - U_0\right) P_0(t) + A_1 P_1(t) + \left(A_2 - \tfrac{1}{3}\omega^2 a^2\right) P_2(t) + \sum_{n=0}^{\infty} A_n P_n(t) = 0$$

$$\cdots (26)$$

This equation is only true for all values of t (or β) if the coefficient of every $P_n(t)$ equals zero. Thus

$$\begin{aligned} A_0 + \tfrac{1}{3}\omega^2 a^2 - U_0 &= 0 \\ A_1 &= 0 \\ A_2 - \tfrac{1}{3}\omega^2 a^2 &= 0 \\ A_3 &= 0, \quad A_4 = 0, \quad \ldots \text{ etc} \end{aligned} \qquad \cdots (27)$$

and the coefficients A_0, A_1, A_2, etc... are determined.

Substituting expressions for A_0, A_1 and A_2 (noting that all other A_n are zero) into (21) gives an expression for the gravitational potential V of a rotationally symmetric ellipsoid

$$V(u,\beta) = \left(U_0 - \tfrac{1}{3}\omega^2 a^2\right) \frac{Q_0\left(i\frac{u}{E}\right)}{Q_0\left(i\frac{b}{E}\right)} + \tfrac{1}{3}\omega^2 a^2 \, \frac{Q_2\left(i\frac{u}{E}\right)}{Q_2\left(i\frac{b}{E}\right)} P_2(t) \qquad \cdots (28)$$

Using the special results of the Legendre functions of the second kind (see p.7)

$$Q_0\left(\frac{iu}{E}\right) = \coth^{-1}\left(\frac{iu}{E}\right) = -i\tan^{-1}\left(\frac{E}{u}\right) \qquad \cdots (29)$$

$$Q_2\left(\frac{iu}{E}\right) = \left[\frac{3}{2}\left(\frac{i^2 u^2}{E^2}\right) - \frac{1}{2}\right]\left[-i\tan^{-1}\left(\frac{E}{u}\right)\right] - \frac{3}{2}\left(\frac{iu}{E}\right)$$

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and since $i^2 = -1$

$$Q_2\left(\frac{iu}{E}\right) = \left(-\frac{3}{2}\frac{u^2}{E^2} - \frac{1}{2}\right)\left(-i \tan^{-1}\frac{E}{u}\right) - \frac{3}{2} i \frac{u}{E}$$

which after some simplification gives

$$Q_2\left(\frac{iu}{E}\right) = \frac{i}{2}\left\{ \left(\frac{3u^2}{E^2} + 1\right)\tan^{-1}\frac{E}{u} - \frac{3u}{E} \right\} \qquad \text{--- (30)}$$

Letting

$$\boxed{q = \frac{1}{2}\left\{\left(1 + \frac{3u^2}{E^2}\right)\tan^{-1}\frac{E}{u} - \frac{3u}{E}\right\}} \qquad \text{--- (31)}$$

and when $u = b$

$$\boxed{q_0 = \frac{1}{2}\left\{\left(1 + \frac{3b^2}{E^2}\right)\tan^{-1}\frac{E}{b} - \frac{3b}{E}\right\}} \qquad \text{--- (32)}$$

Substituting (29), (30) with (31) and (32) into (28) gives

$$V(u,\beta) = \left(U_0 - \frac{1}{3}\omega^2 a^2\right)\frac{\tan^{-1}\frac{E}{u}}{\tan^{-1}\frac{E}{b}} + \frac{1}{3}\omega^2 a^2 \frac{q}{q_0} P_2(t) \qquad \text{--- (33)}$$

Consider the term $\tan^{-1}\frac{E}{u}$ or $\tan^{-1}x$ where $x = \frac{E}{u}$

and since $E = \sqrt{a^2 - u^2}$ the $x < 1$ and $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ then

$$\tan^{-1}\frac{E}{u} = \frac{E}{u} \underbrace{- \frac{(a^2-u^2)^{3/2}}{u^3} + \frac{(a^2-u^2)^{5/2}}{u^5} - \frac{(a^2-u^2)^{7/2}}{u^7} + \ldots\ldots}_{\simeq 0}$$

giving

$$\tan^{-1}\frac{E}{u} \simeq \frac{E}{u} \qquad \text{....(34)}$$

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To find an approximation for $\frac{1}{u}$, the denominator in (34), consider the radial distance r from the centre of the rotational ellipsoid and

$$
\begin{aligned}
r^2 &= X^2 + Y^2 + Z^2 \\
&= (E^2 + u^2)\cos^2\beta\cos^2\lambda + (E^2+u^2)\cos^2\beta\sin^2\lambda + u^2\sin^2\beta \\
&= (E^2 + u^2)\cos^2\beta + u^2\sin^2\beta \\
&= E^2\cos^2\beta + u^2
\end{aligned}
$$

Rewriting as $\quad u^2 = r^2 - E^2\cos^2\beta$

then

$$\frac{1}{u} = (r^2 - E^2\cos^2\beta)^{-\frac{1}{2}} = (a+x)^n$$

Using the BINOMIAL THEOREM

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \cdots\cdots$$

and for $\quad a = r^2, \quad x = E^2\cos^2\beta, \quad n = -\frac{1}{2}$

$$
\begin{aligned}
\frac{1}{u} &= (r^2)^{-\frac{1}{2}} - \tfrac{1}{2}(r^2)^{-\frac{3}{2}}x + \cdots\cdot \\
&= \frac{1}{r} - \frac{x}{2r^3} + \cdots\cdot
\end{aligned}
$$

and for <u>large values of r</u> we can say

$$\frac{1}{u} \simeq \frac{1}{r} \qquad \text{--- (35)}$$

Combining (34) and (35) gives

$$\tan^{-1}\frac{E}{u} \simeq \frac{E}{u} \simeq \frac{E}{r} \qquad \text{--- (36)}$$

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For very large distances r, the 1^{st} term of (33) is dominant and we may write

$$V = \left(U_0 - \frac{1}{3}\omega^2 a^2\right) \frac{E}{\tan^{-1}\frac{E}{b}} \cdot \frac{1}{r} \quad + \quad \text{small terms} \qquad \cdots (37a)$$

and we also know that for very large r, the potential V is

$$V = \frac{GM}{r} + \text{small terms} \qquad \cdots (37b)$$

comparing (37a) and (37b) gives

$$GM = \left(U_0 - \frac{1}{3}\omega^2 a^2\right) \frac{E}{\tan^{-1}\frac{E}{b}} \qquad \cdots (38a)$$

and

$$\boxed{U_0 = \frac{GM}{E} \tan^{-1}\frac{E}{b} + \frac{1}{3}\omega^2 a^2} \qquad \cdots (38b)$$

Equations (38) are the relationships between the mass M of the normal ellipsoid and the potential U_0.

Substituting (38) into (33) gives

$$V(u,\beta) = \frac{GM}{E} \tan^{-1}\frac{E}{u} + \frac{1}{3}\omega^2 a^2 \frac{q}{q_0} P_2(t) \qquad (39)$$

and with
$$P_2(t) = \frac{1}{2}(3t^2 - 1)$$
$$= \frac{3}{2}\sin^2\beta - \frac{1}{2}$$

$$V(u,\beta) = \frac{GM}{E} \tan^{-1}\frac{E}{u} + \frac{1}{2}\omega^2 a^2 \frac{q}{q_0}\left(\sin^2\beta - \frac{1}{3}\right) \qquad \cdots (40)$$

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Adding the potential due to the centrifugal force (22) gives the NORMAL GRAVITY POTENTIAL

$$\boxed{U = \frac{GM}{E}\tan^{-1}\frac{E}{u} + \frac{\omega^2 a^2}{2}\frac{q}{q_o}\left(\sin^2\beta - \frac{1}{3}\right) + \frac{\omega^2}{2}(E^2+u^2)\cos^2\beta} \quad \cdots (41)$$

(Heiskanen & Moritz, 1967, p. 67, eq 2-62)

where GM = universal constant of gravitation G by the mass of the Earth M.

E = focal distance of reference ellipsoid

a = semi-major axis of reference ellipsoid

b = semi-minor " " " "

u = semi-minor axis of confocal ellipsoid passing through point

ω = angular velocity

β = reduced or parametric latitude

$$2q = \left(1 + 3\frac{u^2}{E^2}\right)\tan^{-1}\frac{E}{u} - 3\frac{u}{E}$$

$$2q_o = \left(1 + 3\frac{b^2}{E^2}\right)\tan^{-1}\frac{E}{b} - 3\frac{b}{E}$$

and from Heiskanen & Moritz (1967, p. 228, eq's 6.8)

$$u^2 = (X^2+Y^2+Z^2-E^2)\left\{\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4E^2Z^2}{(X^2+Y^2+Z^2-E^2)^2}}\right\} \quad \cdots (42)$$

$$\tan\beta = \frac{Z\sqrt{E^2+u^2}}{u\sqrt{X^2+Y^2}} \quad \cdots (43)$$

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NORMAL GRAVITY

The normal gravity potential U is a SCALAR FUNCTION of the ORTHOGONAL CURVILINEAR COORDINATES (u, θ, λ), although λ does not appear since the gravity field is rotationally symmetric and thus independent of longitude λ. Nevertheless we may write that U is a function of u, θ, λ or $U(u, \theta, \lambda)$ and that the components of the NORMAL GRAVITY vector $\underset{\sim}{\gamma}$ (gamma) are given by the GRADIENT OF THE GRAVITY POTENTIAL U or

$$\boxed{\underset{\sim}{\gamma} = \text{grad } U = \nabla U = \frac{1}{h_u}\frac{\partial U}{\partial u}\hat{\underset{\sim}{e}}_u + \frac{1}{h_\theta}\frac{\partial U}{\partial \theta}\hat{\underset{\sim}{e}}_\theta + \frac{1}{h_\lambda}\frac{\partial U}{\partial \lambda}\hat{\underset{\sim}{e}}_\lambda} \quad \cdots (44)$$

where

h_u, h_θ, h_λ are <u>scale factors</u>

$\hat{\underset{\sim}{e}}_u, \hat{\underset{\sim}{e}}_\theta, \hat{\underset{\sim}{e}}_\lambda$ are <u>unit vectors</u> in the direction of increasing u, θ or λ

The <u>scale factors</u> h_u, h_θ, h_λ can be determined in the following manner

(i) the cartesian coordinates are (see eq 8)

$$X = \sqrt{E^2 + u^2}\ \sin\theta\ \cos\lambda$$
$$Y = \sqrt{E^2 + u^2}\ \sin\theta\ \sin\lambda$$
$$Z = u \cos\theta \quad \cdots (8)$$

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(ii) the total differentials are

$$dX = \frac{\partial X}{\partial u} \cdot du + \frac{\partial X}{\partial \theta} \cdot d\theta + \frac{\partial X}{\partial \lambda} \cdot d\lambda$$

$$dY = \frac{\partial Y}{\partial u} \cdot du + \frac{\partial Y}{\partial \theta} \cdot d\theta + \frac{\partial Y}{\partial \lambda} \cdot d\lambda$$

$$dZ = \frac{\partial Z}{\partial u} \cdot du + \frac{\partial Z}{\partial \theta} \cdot d\theta + \frac{\partial Z}{\partial \lambda} \cdot d\lambda \quad \ldots (45)$$

and an element of distance on the surface is given by

(iii) $$dS^2 = dX^2 + dY^2 + dZ^2 \quad \ldots (46)$$

and since u, θ, λ are orthogonal coords (as are X, Y, Z)

(iv) $$dS^2 = h_u^2\, du^2 + h_\theta^2\, d\theta^2 + h_\lambda^2\, d\lambda^2 \quad \ldots (47)$$

and (46) and (47) give equivalent elemental distances dS but in terms of different orthogonal coordinate systems. By partially differentiating (8) the total differentials are

$$dX = \frac{u}{\sqrt{E^2+u^2}} \sin\theta \cos\lambda\, du + \sqrt{E^2+u^2} \cos\theta \cos\lambda\, d\theta - \sqrt{E^2+u^2} \sin\theta \sin\lambda\, d\lambda$$

$$dY = \frac{u}{\sqrt{E^2+u^2}} \sin\theta \sin\lambda\, du + \sqrt{E^2+u^2} \cos\theta \sin\lambda\, d\theta + \sqrt{E^2+u^2} \sin\theta \cos\lambda\, d\lambda$$

$$dZ = \cos\theta\, du - u \sin\theta\, d\theta$$

Squaring and gathering terms, noting that terms $du.d\theta$, $du d\lambda$, $d\theta\, d\lambda$ cancel (since u, θ, λ are orthogonal)

THE NORMAL GRAVITY FIELD

$$ds^2 = \left(\frac{E^2\cos^2\theta + u^2}{E^2+u^2}\right) du^2 + (E^2\cos^2\theta + u^2)\, d\theta^2 + (E^2+u^2)\sin^2\theta\, d\lambda^2$$

or

$$ds^2 = W^2 du^2 + W^2(E^2+u^2)\, d\theta^2 + (E^2+u^2)\sin^2\theta\, d\lambda^2 \qquad \ldots (48)$$

and the scale factors are

$$h_u = W = \sqrt{\frac{E^2\cos^2\theta + u^2}{E^2+u^2}}$$

$$h_\theta = W\sqrt{E^2+u^2}$$

$$h_\lambda = \sqrt{E^2+u^2}\,\sin\theta \qquad \ldots (49)$$

Re-writing (44) as

$$\underset{\sim}{\gamma} = \gamma_u\, \underset{\sim}{\hat{e}}_u + \gamma_\theta\, \underset{\sim}{\hat{e}}_\theta + \gamma_\lambda\, \underset{\sim}{\hat{e}}_\lambda \qquad \ldots (50)$$

where

$$\gamma_u = \frac{1}{h_u}\frac{\partial U}{\partial u} = \frac{1}{W}\frac{\partial U}{\partial u}$$

$$\gamma_\theta = \frac{1}{h_\theta}\frac{\partial U}{\partial \theta} = \frac{1}{W\sqrt{E^2+u^2}}\frac{\partial U}{\partial \theta}$$

$$\gamma_\lambda = \frac{1}{h_\lambda}\frac{\partial U}{\partial \lambda} = \frac{1}{\sqrt{E^2+u^2}\,\sin\theta}\frac{\partial U}{\partial \lambda} \qquad \ldots (51)$$

and noting that (44) and (50) describe a VECTOR normal to the ellipsoid surface whose positive direction is <u>OUTWARDS</u> from the surface.

THE NORMAL GRAVITY FIELD

Re-stating (41)

$$U = \frac{GM}{E}\tan^{-1}\frac{E}{u} + \frac{\omega^2 a^2}{2}\frac{q}{q_0}\left(\sin^2\beta - \frac{1}{3}\right) + \frac{\omega^2}{2}\left(E^2+u^2\right)\cos^2\beta$$

noting that GM, E, ω, a, and q_0 are constants and that U is a function of u, β where $\beta = 90 - \theta$ the partial derivatives are

(i) <u>the partial derivative</u> $\frac{\partial U}{\partial u}$

$$\frac{\partial U}{\partial u} = \frac{GM}{E}\frac{\partial}{\partial u}\left(\tan^{-1}\frac{E}{u}\right) + \frac{\omega^2 a^2}{q_0}\left(\frac{1}{2}\sin^2\beta - \frac{1}{6}\right)\frac{\partial q}{\partial u} + \omega^2 u \cos^2\beta$$

where

$$\frac{\partial}{\partial u}\left(\tan^{-1}\frac{E}{u}\right) = \frac{-E}{E^2+u^2}$$

and from (31)

$$\frac{\partial q}{\partial u} = \left(\frac{1}{2} + \frac{3u^2}{2E^2}\right)\left(\frac{-E}{E^2+u^2}\right) + \frac{3u}{E^2}\tan^{-1}\frac{E}{u} - \frac{3}{2E}$$

and

$$q' = -\left(\frac{E^2+u^2}{E}\right)\frac{\partial q}{\partial u} = 3\left(1+\frac{u^2}{E^2}\right)\left(1 - \frac{u}{E}\tan^{-1}\frac{E}{u}\right) - 1 \qquad \cdots(52)$$

gives

$$-\frac{\partial U}{\partial u} = \frac{GM}{E^2+u^2} + \frac{E}{E^2+u^2}\frac{q'}{q_0}\,\omega^2 a^2\left(\frac{1}{2}\sin^2\beta - \frac{1}{6}\right) - \omega^2 u\cos^2\beta \qquad \cdots(53)$$

THE NORMAL GRAVITY FIELD

(ii) the partial derivative $\dfrac{\partial U}{\partial \beta}$

$$\frac{\partial U}{\partial \beta} = \omega^2 a^2 \frac{q}{q_0} \sin\beta \cos\beta - \omega^2 E^2 \cos\beta \sin\beta - \omega^2 u^2 \cos\beta \sin\beta$$

or

$$-\frac{\partial U}{\partial \beta} = \left\{ -\omega^2 a^2 \frac{q}{q_0} + \omega^2 (E^2 + u^2) \right\} \sin\beta \cos\beta \qquad \text{--- (54)}$$

(iii) the partial derivative $\dfrac{\partial U}{\partial \lambda} = 0$ (rotational symmetry)

Substituting (54) and (53) into (51) gives

$$-w\gamma_u = \frac{GM}{E^2 + u^2} + \frac{\omega^2 a^2 E}{E^2 + u^2} \frac{q'}{q_0} \left(\frac{1}{2} \sin^2\beta - \frac{1}{6} \right) - \omega^2 u \cos^2\beta \qquad \text{...(55)}$$

$$-w\gamma_\beta = \left\{ \frac{-\omega^2 a^2}{\sqrt{E^2 + u^2}} \frac{q}{q_0} + \omega^2 \sqrt{E^2 + u^2} \right\} \sin\beta \cos\beta \qquad \text{...(56)}$$

where q and q_0 are given by (31) and (32)
q' is given by (52)

$$w = \sqrt{\frac{E^2 \sin^2\beta + u^2}{E^2 + u^2}} \qquad \text{(see (49) with } \cos\theta = \sin\beta\text{)}$$

Equations (55) & (56) are the same as Heiskanen & Moritz 1967, equ's 2-66, 2-67 p. 68.

NOTE the NEGATIVE sign in (55) and (56) indicates components of the GRAVITY VECTOR or INWARD NORMAL

THE NORMAL GRAVITY FIELD

For the LEVEL ELLIPSOID of surface S_o and POTENTIAL U_o

$u = b$

In this case $\frac{q}{q_o} = 1$ (see equ's (31) and (32))

and also $\sqrt{E^2 + u^2} = \sqrt{E^2 + b^2} = a$

substituting these into (56) gives

$$\gamma_{\beta,o} = 0$$

surface S_o

and also because of rotational symmetry

$$\gamma_\lambda = 0$$

Hence, the magnitude of gravity on the LEVEL ELLIPSOID S_o (denoted by γ) is given by

$\gamma = \gamma_{u,o}$ where $u = b$ in (55) or

$$W_o \, \gamma_{u,o} = \frac{GM}{E^2+b^2} + \frac{\omega^2 a^2 E}{E^2+b^2} \frac{q_o'}{q_o}\left(\frac{1}{2}\sin^2\beta - \frac{1}{6}\right) - \omega^2 b \cos^2\beta$$

and since $E^2 + b^2 = a^2$

$$W_o \, \gamma_{u,o} = \frac{GM}{a^2}\left\{ 1 + \frac{\omega^2 a^2 E}{GM} \frac{q_o'}{q_o}\left(\frac{1}{2}\sin^2\beta - \frac{1}{6}\right) - \frac{\omega^2 a^2 b}{GM}\cos^2\beta \right\} \quad \text{...(57)}$$

where

$$W_o = \frac{\sqrt{a^2 \sin^2\beta + b^2 \cos^2\beta}}{a} \quad \text{...(58)}$$

(see eq on p.20 with $u = b$ and $E^2 = a^2 - b^2$)

THE NORMAL GRAVITY FIELD

The substitutions $m = \dfrac{\omega^2 a^2 b}{GM}$ ... (59)

$$e' = \frac{E}{b} = \frac{\sqrt{a^2-b^2}}{b} \quad \text{... (60)}$$

(where e' is the SECOND ECCENTRICITY of the ellipsoid)

can be made to simplify (57) to read

$$\gamma = \frac{GM}{a\sqrt{a^2\sin^2\beta + b^2\cos^2\beta}}\left\{1 + me'\frac{q_0'}{q_0}\left(\frac{1}{2}\sin^2\beta - \frac{1}{6}\right) - m\cos^2\beta\right\}$$

Letting $x = \dfrac{me'q_0'}{q_0}$, the term in brackets $\{\ \}$ becomes

$$1 + \frac{x}{2}\sin^2\beta - \frac{x}{6} - m\cos^2\beta$$

$$1 + \frac{x}{2} - \frac{x}{2}\cos^2\beta - \frac{x}{6} - m\cos^2\beta \qquad (\text{since } \sin^2\beta = 1-\cos^2\beta)$$

$$1 + \frac{x}{3} - \frac{x}{2}\cos^2\beta - m\cos^2\beta$$

and since $\cos^2\beta + \sin^2\beta = 1$

$$\cos^2\beta + \sin^2\beta + \frac{x}{3}\cos^2\beta + \frac{x}{3}\sin^2\beta - \frac{x}{2}\cos^2\beta - m\cos^2\beta$$

giving

$$\left(1 + \frac{x}{3}\right)\sin^2\beta + \left(1 - m - \frac{x}{6}\right)\cos^2\beta$$

and

$$\gamma = \frac{GM}{a\sqrt{a^2\sin^2\beta + b^2\cos^2\beta}}\left\{\left(1 + \frac{me'}{3}\frac{q_0'}{q_0}\right)\sin^2\beta + \left(1 - m - \frac{me'}{6}\frac{q_0'}{q_0}\right)\cos^2\beta\right\} \quad \text{... (61)}$$

(same as Heiskanen & Moritz, 1967, eq 2-72, p69)

THE NORMAL GRAVITY FIELD

AT THE EQUATOR ($\beta = 0°$) $\sin\beta = 0$, $\cos\beta = 1$

$$\gamma_e = \frac{GM}{ab}\left(1 - m - \frac{me'}{6}\frac{q_0'}{q_0}\right) \qquad \text{--- (62 a)}$$

↑ equator

AT THE POLES ($\beta = \pm 90°$) $\sin\beta = 1$, $\cos\beta = 0$

$$\gamma_p = \frac{GM}{a^2}\left(1 + \frac{me'}{3}\frac{q_0'}{q_0}\right) \qquad \text{--- (62b)}$$

↑ pole

and $\gamma_p > \gamma_e$

Now the FLATTENING of the ellipsoid is f where

$$f = \frac{a-b}{a} \qquad \text{--- (63 a)}$$

and introducing the GRAVITY FLATTENING f^* as

$$f^* = \frac{\gamma_p - \gamma_e}{\gamma_e} \qquad \text{--- (63b)}$$

then $$f + f^* = \frac{a\gamma_p - b\gamma_e}{a\gamma_e} \qquad \text{--- (64)}$$

Substituting (62a), (62b) into (64) with $x = me'\frac{q_0'}{q_0}$, $m = \frac{\omega^2 a^2 b}{GM}$

the R.H.S. of (64) becomes

$$\begin{aligned} RHS &= \frac{b}{\gamma_e}\left(\frac{\gamma_p}{b} - \frac{\gamma_e}{a}\right) \\ &= \frac{b}{\gamma_e}\left\{\frac{GM}{a^2 b}\left(1 + \frac{x}{3}\right) - \frac{GM}{a^2 b}\left(1 - m - \frac{x}{6}\right)\right\} \end{aligned}$$

THE NORMAL GRAVITY FIELD

$$= \frac{b}{\gamma_e}\left\{ \frac{GM}{a^2 b}\left(m + \frac{x}{2} \right) \right\}$$

$$= \frac{b}{\gamma_e}\left\{ \frac{GM}{a^2 b}\, m \left(1 + \frac{e' q_0'}{2 q_0} \right) \right\}$$

$$= \frac{\omega^2 b}{\gamma_e}\left(1 + \frac{e' q_0'}{2 q_0} \right)$$

hence

$$\boxed{f + f^* = \frac{\omega^2 b}{\gamma_e}\left(1 + \frac{e' q_0'}{2 q_0} \right)} \qquad \text{--- (65)}$$

This is the rigorous form of CLAIRAUT'S EQUATION originally developed in an approximate form by Alexis Claude Clairaut (1713 - 65) in his "Théorie de la figure de la terre" in 1743 (Theory of the figure of the Earth). In its original form, Clairaut's equation demonstrated that the flattening of the Earth can be obtained from gravity observations.

By a re-arrangement of (62a) and (62b)

$$\frac{ab}{GM}\gamma_e = 1 - m - \frac{me'}{6}\frac{q_0'}{q_0}$$

and

$$\frac{a^2 \gamma_p}{GM} = 1 + \frac{me'}{3}\frac{q_0'}{q_0}$$

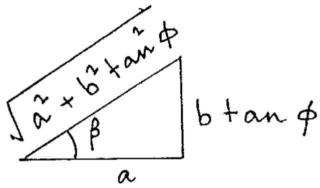
which can be substituted into (61) to give the equation for NORMAL GRAVITY γ as

$$\boxed{\gamma = \frac{a\gamma_p \sin^2\beta + b\gamma_e \cos^2\beta}{\sqrt{a^2\sin^2\beta + b^2\cos^2\beta}}} \qquad \text{--- (66)}$$

THE NORMAL GRAVITY FIELD

Rather than using the REDUCED LATITUDE β, the GEODETIC LATITUDE ϕ may be used. From considerations of geometric geodesy, reduced and geodetic latitudes are related by

$$\tan\beta = \frac{b}{a}\tan\phi$$



and from diagram

$$\sin^2\beta = \frac{b^2\tan^2\phi}{a^2 + b^2\tan^2\phi}$$

and multiplying numerator and denominator by $\cos^2\phi$ gives

$$\sin^2\beta = \frac{b^2\sin^2\phi}{a^2\cos^2\phi + b^2\sin^2\phi}$$

and similarly, from diagram

$$\cos^2\beta = \frac{a^2\cos^2\phi}{a^2\cos^2\phi + b^2\sin^2\phi}$$

giving

$$a^2\sin^2\beta + b^2\cos^2\beta = \frac{a^2b^2}{a^2\cos^2\phi + b^2\sin^2\phi}$$

and

$$a\gamma_p\sin^2\beta + b\gamma_e\cos^2\beta = \frac{ab(\gamma_p b\sin^2\phi + \gamma_e a\cos^2\phi)}{a^2\cos^2\phi + b^2\sin^2\phi}$$

and substituting into (66) gives SOMIGLIANA'S FORMULA for NORMAL GRAVITY

$$\boxed{\gamma = \frac{b\gamma_p\sin^2\phi + a\gamma_e\cos^2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}} \qquad \text{--- (67)}$$

THE NORMAL GRAVITY FIELD

Equation (67) can be modified by the following

the numerator $a\gamma_e\left(\cos^2\phi + \dfrac{b\gamma_p}{a\gamma_e}\sin^2\phi\right)$

$$= a\gamma_e\left(1-\sin^2\phi + \frac{b\gamma_p}{a\gamma_e}\sin^2\phi\right)$$

$$= a\gamma_e\left(1+\sin^2\phi\left(\frac{b\gamma_p}{a\gamma_e}-1\right)\right)$$

$$= a\gamma_e\left(1+k\sin^2\phi\right) \quad \text{where} \quad k = \frac{b\gamma_p}{a\gamma_e}-1$$

the denominator $= \sqrt{a^2\cos^2\phi + b^2\sin^2\phi}$

$$= \sqrt{a^2\left(\cos^2\phi + \frac{b^2}{a^2}\sin^2\phi\right)}$$

$$= a\sqrt{1-\sin^2\phi + \frac{b^2}{a^2}\sin^2\phi}$$

$$= a\sqrt{1-\left(\frac{a^2-b^2}{a^2}\right)\sin^2\phi}$$

$$= a\sqrt{1-e^2\sin^2\phi} \quad \text{where} \quad e^2 = \frac{a^2-b^2}{a^2}$$

and (67) becomes

$$\boxed{\gamma = \gamma_e\frac{1+k\sin^2\phi}{\sqrt{1-e^2\sin^2\phi}}} \quad \text{.... (68)}$$

where $k = \dfrac{b\gamma_p}{a\gamma_e} - 1$

$e^2 = \dfrac{a^2-b^2}{a^2}$

Eq (68) is known as PIZETTI'S EQUATION
(see Pick, et al, p. 49, eq 142)

THE NORMAL GRAVITY FIELD

SPHERICAL HARMONIC EXPANSION OF NORMAL POTENTIAL.

The GRAVITATIONAL POTENTIAL V of the MODEL EARTH (a rotational equipotential ellipsoid) in terms of ELLIPSOIDAL COORDINATES (u, β, λ) is given by (39) as

$$V(u,\beta) = \frac{GM}{E}\tan^{-1}\frac{E}{u} + \frac{1}{3}\omega^2 a^2 \frac{q}{q_0} P_2(t) \qquad \cdots (39)$$

where $t = \sin\beta$ and β is the reduced latitude.
(Page 15 has a summary of the other variables)

To develop the SPHERICAL HARMONIC expansion of the normal gravitational potential V consider the relationship between cartesian coords (X, Y, Z) SPHERICAL COORDS (r, θ, λ) and ELLIPSOIDAL COORDS (u, β, λ)

$$\begin{aligned} X &= r\sin\theta\cos\lambda &&= \sqrt{E^2+u^2}\cos\beta\cos\lambda \\ Y &= r\sin\theta\sin\lambda &&= \sqrt{E^2+u^2}\cos\beta\sin\lambda \\ Z &= r\cos\theta &&= u\sin\beta \end{aligned} \qquad \cdots\ (69)$$

Now

$$\frac{Z}{Y} = \frac{r\cos\theta}{r\sin\theta\sin\lambda} = \frac{u\sin\beta}{\sqrt{E^2+u^2}\cos\beta\sin\lambda}$$

giving

$$\cot\theta = \frac{u}{\sqrt{E^2+u^2}}\tan\beta \qquad \cdots (70)$$

and also

$$X^2+Y^2+Z^2 = r^2 = E^2\cos^2\beta + u^2 \qquad \text{(see p. 13)}$$

giving

$$r = \sqrt{E^2\cos^2\beta + u^2} \qquad \cdots (71)$$

THE NORMAL GRAVITY FIELD

Now, in (39) $\tan^{-1}\frac{E}{u}$ may be expanded using the series

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots \quad \text{for } |x| < 1$$

hence

$$\tan^{-1}\frac{E}{u} = \frac{E}{u} - \frac{1}{3}\left(\frac{E}{u}\right)^3 + \frac{1}{5}\left(\frac{E}{u}\right)^5 - \frac{1}{7}\left(\frac{E}{u}\right)^7 + \dots \qquad (72)$$

In (39) the variable q is given by (31) as

$$q = \frac{1}{2}\left\{\left(1 + 3\left(\frac{u}{E}\right)^2\right)\tan^{-1}\frac{E}{u} - 3\frac{u}{E}\right\}$$

and substituting (72) gives

$$q = \frac{1}{2}\left\{\left(1 + 3\left(\frac{u}{E}\right)^2\right)\left(\frac{E}{u} - \frac{1}{3}\left(\frac{E}{u}\right)^3 + \frac{1}{5}\left(\frac{E}{u}\right)^5 - \frac{1}{7}\left(\frac{E}{u}\right)^7 + \dots\right) - 3\frac{u}{E}\right\}$$

$$= \frac{1}{2}\left\{\frac{E}{u} - \frac{1}{3}\left(\frac{E}{u}\right)^3 + \frac{1}{5}\left(\frac{E}{u}\right)^5 - \frac{1}{7}\left(\frac{E}{u}\right)^7 + \dots\right.$$

$$\left. + 3\frac{u}{E} - \frac{E}{u} + \frac{3}{5}\left(\frac{E}{u}\right)^3 - \frac{3}{7}\left(\frac{E}{u}\right)^5 + \frac{3}{9}\left(\frac{E}{u}\right)^7 - \dots - 3\frac{u}{E}\right\}$$

$$= \frac{1}{2}\left\{\frac{4}{15}\left(\frac{E}{u}\right)^3 - \frac{8}{35}\left(\frac{E}{u}\right)^5 + \frac{12}{63}\left(\frac{E}{u}\right)^7 - \dots\right\}$$

and

$$q = \frac{2}{3.5}\left(\frac{E}{u}\right)^3 - \frac{4}{7.5}\left(\frac{E}{u}\right)^5 + \frac{6}{7.9}\left(\frac{E}{u}\right)^7 - \dots$$

which can be expressed in the form

$$q = -\sum_{n=1}^{\infty}(-1)^n \frac{2n}{(2n+1)(2n+3)}\left(\frac{E}{u}\right)^{2n+1} \qquad \dots (73)$$

and

$$\tan^{-1}\frac{E}{u} = \left(\frac{E}{u}\right) + \sum_{n=1}^{\infty}(-1)^n \frac{1}{2n+1}\left(\frac{E}{u}\right)^{2n+1} \qquad \dots (74)$$

THE NORMAL GRAVITY FIELD

Using (73) and (74) the expression for the normal gravitational potential V becomes

$$V = \frac{GM}{E}\left\{\frac{E}{u} + \sum_{n=1}^{\infty}(-1)^n \frac{1}{2n+1}\left(\frac{E}{u}\right)^{2n+1}\right\} - \frac{\omega^2 a^2}{3q_0}\sum_{n=1}^{\infty}(-1)^n \frac{2n}{(2n+1)(2n+3)}\left(\frac{E}{u}\right)^{2n+1} P_2(t)$$

$$= \frac{GM}{u} + \frac{GM}{E}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{E}{u}\right)^{2n+1} - \frac{\omega^2 a^2}{3q_0}\sum_{n=1}^{\infty}(-1)^n \frac{2n}{(2n+1)(2n+3)}\left(\frac{E}{u}\right)^{2n+1} P_2(t)$$

and with the substitutions $m = \dfrac{\omega^2 a^2 b}{GM}$

$$e' = \frac{E}{b}$$

then $me' = \dfrac{\omega^2 a^2 E}{GM}$

and

$$V = \frac{GM}{u} + \frac{GM}{E}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{E}{u}\right)^{2n+1} - \frac{GM}{E}\left\{\frac{me'}{3q_0}\sum_{n=1}^{\infty}(-1)^n \frac{2n}{(2n+1)(2n+3)}\left(\frac{E}{u}\right)^{2n+1}\right\} P_2(t)$$

$$= \frac{GM}{u} + \sum_{n=1}^{\infty}(-1)^n \frac{GM}{(2n+1)E}\left(\frac{E}{u}\right)^{2n+1} - \left\{\sum_{n=1}^{\infty}(-1)^n \frac{GM}{E}\frac{2n}{(2n+1)(2n+3)}\left(\frac{E}{u}\right)^{2n+1}\cdot\frac{me'}{3q_0}\right\} P_2(t)$$

and

$$\boxed{V = \frac{GM}{u} + \sum_{n=1}^{\infty}(-1)^n \frac{GM}{E}\frac{1}{2n+1}\left(\frac{E}{u}\right)^{2n+1}\left\{1 - \frac{me'}{3q_0}\frac{2n}{2n+3}P_2(t)\right\}} \quad \text{---(75)}$$

$$t = \sin\beta$$

(same as Heiskanen & Moritz, 1967, p.72, eq 2-87)

THE NORMAL GRAVITY FIELD

Now consider the spherical harmonic expansion for V developed in the notes "THE GRAVITY FIELD OF THE EARTH" (see p. 44, eq. 71)

$$V = \frac{GM}{r}\left[1 + \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_n^m(t)\left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right)\right]$$

where $t = \cos\theta$, θ is geocentric co-latitude

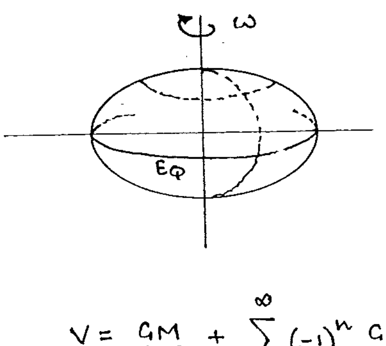
For rotational symmetry, all longitude terms must be zero, hence $m=0$. For symmetry about the equatorial plane only even zonal harmonic terms are permissable. Hence

$$V = \frac{GM}{r}\left[1 + \sum_{n=2,4,6,\ldots\text{ even}}^{\infty}\left(\frac{a}{r}\right)^n P_n(t)\right] \quad \text{.... (76)}$$

or expanded

$$V = \frac{GM}{r} + A_2\frac{P_2(t)}{r^3} + A_4\frac{P_4(t)}{r^5} + A_6\frac{P_6(t)}{r^7} + \cdots \quad \text{... (77)}$$

To determine the CONSTANTS $A_2, A_4, A_6, \cdots$ in (77) choose a point on the axis of rotation of the ellipsoid ($\beta = 90°$ or $\theta = 0°$) and outside the ellipsoid, hence by (71), since $\beta = 90°$ and $\cos\beta = 0$, $u = r$



then equation (75) becomes with $t = \sin\beta$ and $P_2(t) = \frac{1}{2}(3\sin^2\beta - 1)$ gives $P_2(t) = 1$ when $\beta = 90°$

$$V = \frac{GM}{r} + \sum_{n=1}^{\infty}(-1)^n \frac{GM}{E}\frac{1}{2n+1}\frac{E^{2n+1}}{r^{2n+1}}\left\{1 - \frac{me'}{3q_0}\frac{2n}{2n+3}\right\}$$

THE NORMAL GRAVITY FIELD

and

$$V = \frac{GM}{r} + \sum_{n=1}^{\infty} (-1)^n \frac{GM\,E^{2n}}{2n+1} \left\{ 1 - \frac{2n}{2n+3} \frac{me'}{3q_0} \right\} \frac{1}{r^{2n+1}} \qquad \text{---- (78)}$$

For $n = 1, 2, 3, \ldots$ equations (77) and (78) can be compared to give expressions for the constants A_2, A_4, A_6 etc..

$n = 1$ $$\frac{A_2}{r^3} P_2(\cos\theta) = -\frac{GM \cdot E^2}{3} \left\{ 1 - \frac{2}{5} \frac{me'}{3q_0} \right\} \frac{1}{r^3}$$

$n = 2$ $$\frac{A_4}{r^5} P_4(\cos\theta) = \frac{GM \cdot E^4}{5} \left\{ 1 - \frac{4}{7} \frac{me'}{3q_0} \right\} \frac{1}{r^5}$$

$n = 3$ $$\frac{A_6}{r^7} P_6(\cos\theta) = -\frac{GM \cdot E^6}{7} \left\{ 1 - \frac{6}{9} \frac{me'}{3q_0} \right\} \frac{1}{r^7}$$

and since $\theta = 0°$, $\cos\theta = 1$

$$P_2(\cos\theta) = 1$$
$$P_4(\cos\theta) = 1$$
$$P_6(\cos\theta) = 1$$
$$\vdots$$
$$P_n(\cos\theta) = 1 \quad \text{when } \theta = 0°$$

hence

$$A_2 = -\frac{GM \cdot E^2}{3} \left(1 - \frac{2}{5} \frac{me'}{3q_0} \right)$$

$$A_4 = \frac{GM \cdot E^4}{5} \left(1 - \frac{4}{7} \frac{me'}{3q_0} \right)$$

$$A_6 = -\frac{GM \cdot E^6}{7} \left(1 - \frac{6}{9} \frac{me'}{3q_0} \right)$$

$$\vdots$$

or

$$A_{2n} = (-1)^n \frac{GM \cdot E^{2n}}{2n+1} \left(1 - \frac{2n}{2n+3} \frac{me'}{3q_0} \right) \qquad \text{.... (79)}$$

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Eq (77) with the coefficients determined by (79) give the gravitational potential of an equipotential ellipsoid as a series of spherical harmonics.

It is known that the second degree coefficient A_2 is

$$A_2 = G(A - C)$$

where C = moment of inertia w.r.t. axis of revolution
A = moment of inertia w.r.t. any axis in the equatorial plane (see Heiskanen & Moritz, 1967, pp. 61-63 and pp. 72-73)

and also from (79)

$$A_2 = -\frac{GM \cdot E^2}{3}\left(1 - \frac{2}{5}\frac{me'}{3q_0}\right)$$

hence

$$G(C - A) = \frac{GM \cdot E^2}{3}\left(1 - \frac{2}{15}\frac{me'}{q_0}\right) \qquad \ldots (80)$$

Eq (80) shows that the principal moments of inertia can be expressed in terms of the constants a, b, M & ω remembering that $m = \frac{\omega^2 a^2 b}{GM}$, $e' = \frac{E}{b}$ and q_0 is a

function of E and b only (see eq. 32)

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It is possible to determine another expression for the term $\frac{me'}{3q_0}$ in equation (79) by manipulating (80) as follows

$$G(C-A) = \frac{GM \cdot E^2}{3}\left(1 - \frac{2}{15}\frac{me'}{q_0}\right)$$

$$\frac{(C-A)}{M \cdot E^2} = \frac{1}{3} - \frac{2}{15}\frac{me'}{3q_0}$$

and $$\frac{me'}{3q_0} = \frac{5}{2} - \frac{15(C-A)}{2M \cdot E^2}$$

Now, substituting into (79) gives

$$A_{2n} = (-1)^n \frac{GM \cdot E^{2n}}{2n+1}\left(1 - \frac{2n}{2n+3}\left(\frac{5}{2} - \frac{15(C-A)}{2M \cdot E^2}\right)\right)$$

$$= (-1)^n \frac{GM \cdot E^{2n}}{2n+1}\left\{\frac{2n+3}{2n+3} - \frac{2n}{2n+3}\left(\frac{5}{2} - \frac{15(C-A)}{2M \cdot E^2}\right)\right\}$$

$$= (-1)^n \frac{GM \cdot E^{2n}}{(2n+1)(2n+3)}\left\{2n+3 - 5n + \frac{15n(C-A)}{M \cdot E^2}\right\}$$

and

$$A_{2n} = (-1)^n \frac{3GM \cdot E^{2n}}{(2n+1)(2n+3)}\left\{1 - n + \frac{5n(C-A)}{M \cdot E^2}\right\} \quad \ldots\ldots (81)$$

It is desirable to express the potential V in the form

$$\boxed{V = \frac{GM}{r}\left\{1 - \sum_{n=1}^{\infty} J_{2n}\left(\frac{a}{r}\right)^{2n} P_{2n}(t)\right\}} \quad \ldots (82)$$

where $t = \cos\theta$

which can be exanded into the following form

$$V = \frac{GM}{r} - J_2\frac{GM}{r}\left(\frac{a}{r}\right)^2 P_2(t) - J_4\frac{GM}{r}\left(\frac{a}{r}\right)^4 P_4(t) - \ldots\ldots \quad (83)$$

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By comparing terms of (83) and (77), the general relationship follows

$$J_{2n} = -\frac{A_{2n}}{GM \cdot a^{2n}} \qquad n = 1, 2, 3, \ldots$$

and from (81)

$$J_{2n} = (-1)^{n+1} \frac{3}{(2n+1)(2n+3)} \left(\frac{E}{a}\right)^{2n} \left\{ 1 - n + \frac{5n(C-A)}{M \cdot E^2} \right\} \qquad \ldots (84)$$

Now by definition $e = \frac{E}{a}$ $\quad e =$ eccentricity

and for $n = 1$

$$J_2 = \frac{3e^2}{3 \cdot 5} \left\{ 1 - 1 + \frac{5(C-A)}{M \cdot E^2} \right\}$$

$$\boxed{J_2 = \frac{e^2(C-A)}{M \cdot E^2} = \frac{C-A}{Ma^2}} \qquad \ldots (85)$$

hence $\frac{J_2}{e^2} = \frac{C-A}{M \cdot E^2}$ which when substituted into (84) gives the J_{2n} coefficients in terms of J_2

$$\boxed{J_{2n} = (-1)^{n+1} \frac{3e^{2n}}{(2n+1)(2n+3)} \left\{ 1 - n + 5n \frac{J_2}{e^2} \right\}} \qquad \ldots (86)$$

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Comparing equations (85) and (80) gives an expression for J_2

$$J_2 = \frac{e^2}{3}\left(1 - \frac{2}{15}\frac{me'}{q_0}\right) \qquad \cdots (87)$$

and with $e = \frac{E}{a}$, $e' = \frac{E}{b}$ and $e' = \frac{e}{\sqrt{1-e^2}}$

then $be' = ea$ and with $m = \frac{\omega^2 a^2 b}{GM}$

then $me' = \frac{\omega^2 a^3 e}{GM}$

which inserted into (87) and rearranged gives an equation from which e^2 may be computed

$$\boxed{e^2 = 3J_2 + \frac{4}{15}\frac{\omega^2 a^3}{GM}\frac{e^3}{2q_0}} \qquad \cdots (88)$$

where, from (32) with $e' = \frac{E}{b}$

$$2q_0 = (1 + 3e'^2)\tan^{-1} e' - \frac{3}{e'} \qquad \cdots (89)$$

Equation (88) is the basic equation relating e^2 to a, GM, J_2 and ω and <u>must be solved iteratively for e^2</u>

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Summary An equipotential ellipsoid of revolution can be used as a reference surface for gravity. Such an ellipsoid can be defined by four constants : GEODETIC REFERENCE SYSTEM 1980 (GRS 80)

- $a = 6378137$ m equatorial radius of Earth
- $GM = 3986005 \times 10^{8}$ m^3/s^2 geocentric gravitational constant of the Earth – including the atmosphere
- $J_2 = 108263 \times 10^{-8}$ dynamical form factor
- $\omega = 7292115 \times 10^{-11}$ rad/s angular velocity of Earth.

These are PHYSICAL constants (Ref. Bulletin Geodesique, 1980, Vol. 54, N° 3) from which GEOMETRIC constants

e^2 eccentricity squared can be computed using iteration and (88) and then the following

$b = a\sqrt{1-e^2}$ semi-minor axis

$f = \dfrac{a-b}{a}$ flattening

$E = \dfrac{e}{a}$ linear eccentricity (focal dist.)

Other PHYSICAL constants are :

the normal gravity potential on the surface of the reference ellipsoid U_0

$$U_0 = \frac{GM}{E} \tan^{-1}\frac{E}{b} + \frac{1}{3}\omega^2 a^2 \qquad \ldots (38b)$$

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the normal gravitational potential V

$$V = \frac{GM}{r}\left\{1 - \sum_{n=1}^{\infty} J_{2n}\left(\frac{a}{r}\right)^{2n} P_{2n}(t)\right\} \qquad n = 1, 2, 3, \ldots \infty \qquad \ldots (82)$$

where

$$t = \cos\theta$$

$$J_{2n} = (-1)^{n+1}\frac{3e^{2n}}{(2n+1)(2n+3)}\left\{1 - n + 5n\frac{J_2}{e^2}\right\} \qquad \ldots (86)$$

Normal gravity at the equator γ_e and normal gravity at the poles γ_p are given by

$$\gamma_e = \frac{GM}{ab}\left(1 - m - \frac{me'}{6}\frac{q'_0}{q_0}\right) \qquad \ldots (62a)$$

$$\gamma_p = \frac{GM}{a^2}\left(1 + \frac{me'}{3}\frac{q'_0}{q_0}\right) \qquad \ldots (62b)$$

where

$$q'_0 = 3\left(1 + \left(\frac{b}{E}\right)^2\right)\left(1 - \frac{b}{E}\tan^{-1}\frac{E}{b}\right) - 1 \qquad \ldots (52)$$

$$2q_0 = \left(1 + 3\left(\frac{b}{E}\right)^2\right)\tan^{-1}\frac{E}{b} - 3\frac{b}{E} \qquad \ldots (32)$$

$$m = \frac{\omega^2 a^2 b}{GM}$$

$$e' = \frac{E}{b}$$

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and normal gravity γ can be computed from Somigliana's formula

$$\gamma = \frac{a\gamma_e \cos^2\phi + b\gamma_p \sin^2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} \qquad \ldots\ldots (67)$$

or from Pizetti's equation

$$\gamma = \gamma_e \frac{1 + k\sin^2\phi}{\sqrt{1-e^2\sin^2\phi}} \qquad \ldots\ldots (68)$$

where $k = \dfrac{b\gamma_p}{a\gamma_e} - 1$

REFERENCES

Heiskanen, W.A. & Moritz, H. (1967). "Physical Geodesy", W.H. Freeman & Co., London. Chapters 1, 2 and 6.

Pick, M., Picha, J. and Vyskocil, V. (1973). "Theory of the Earth's Gravity Field." Elsevier Scientific Publishing Co. London.

Bulletin Geodesique, 1980, Vol. 54, No. 3.